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Upstream influence in boundary layers 45 years ago

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My two-part paper 'Boundary layers and upstream influence', published in 1953, surveyed a wide range of experimental evidence on how a disturbance in supersonic flow, which in inviscid theory would affect only downstream conditions, is able to exercise an upstream influence through the agency of a boundary layer, either laminar or turbulent. Then, by systematically comparing the data with existing attempts to account for the phenomenon theoretically, it concluded that, essentially, two mechanisms of upstream influence exist.

Mechanism (i), first suggested by Oswatitsch & Wieghardt in 1941, depended on a particular property of supersonic flow over a wall: that either wall curvature on inviscid theory, or (for a flat wall) curvature $\mathrm{d}^2\delta_1/\mathrm{d}x^2$ of the displacement-thickness contour on boundary-layer theory, generates a proportional pressure gradient; which, in the latter case, is $A_2\mathrm{d}^2\delta_1/\mathrm{d}x^2$, A_2 being a known positive function of Mach number. Also, this positive pressure gradient might be expected to thicken the layer at a spatial rate $\mathrm{d}\delta_1/\mathrm{d}x = A_1(A_2\mathrm{d}^2\delta_1/\mathrm{d}x^2)$, where A_1, although far from precisely known, must be less for a turbulent than for a laminar layer; so that, finally, the e-folding distance of upstream influence would be A_1A_2.

Mechanism (i) was compared, in part II of my paper, with a different proposal (see the work of Howarth in 1948) for a theoretical programme concerned with 'propagation up the subsonic layer', in which only the undisturbed boundary-layer distribution (including its subsonic part) would be taken, as influenced by viscosity, while disturbances to it would be treated inviscidly. The reason why attempts to carry out this programme had failed was explained in terms of earlier theories of boundary-layer instability, in which time-dependent disturbances had been found to be influenced by viscosity in two layers: a wall layer and a critical layer. For disturbances independent of time these would coincide into a single wall layer in which, however, the influence of viscosity still needed to be taken into account; in which case, the analysis could be satisfactorily completed but became in essence merely an expression of mechanism (i) with a relatively precise determination of A_1.

Mechanism (ii), identified in work by Lees in 1949 at Princeton and by Liepmann, Roshko & Dhawan in 1949 at Caltech, depended on the upstream spreading of a separation bubble till it became sufficiently slender to cause no further separation ahead of it. Part I of my paper was concerned to point out that, although mechanism (i) can work only when a well-defined coefficient A_2 exists (that is, for supersonic flow), mechanism (ii) is effective in both subsonic and supersonic flow. This was illustrated by analysing data on flow up a step at various Mach numbers (with various locations for transition to turbulence) in terms of boundary-layer separation studies. Those instructive examples, which may today be somewhat less known, and which included several interesting cases of both steady and also unsteady separated flows, can appropriately be recalled in a colloquium devoted to such phenomena.

Keywords: boundary layers; upstream influence; separated flows; supersonic flow; subsonic flow

1. Introduction

For this European Mechanics Colloquium held in Manchester University, I was invited to look back 45 years to 1953, when my paper 'Boundary layers and upstream influence' appeared (Lighthill 1953*a*, *b*) in two parts, describing studies which I had pursued in Manchester's Mathematics Department from October 1949 to October 1952 (the paper's date of submission), and relating them both to other theoretical studies and also to extensive wind-tunnel work carried out during the same period in various centres, including the fine Fluid Motion Laboratory created at Manchester by S. Goldstein and directed by W. A. Mair. After Goldstein resigned from the Beyer Chair of Applied Mathematics in order to move from Manchester to Haifa, I applied for that chair; and I remember speaking colourfully, during my interview (April 1950), about much of the research in which I was then active. In particular, I remember describing the upstream-influence phenomena that I was studying as 'an intriguing departure from St Venant's Principle' (the principle that, in general, detailed influences of a local disturbance on a thin plate or in a thin layer do not penetrate over a distance of many thicknesses).

(*a*) *A golden age (1949–1952) of upstream-influence research*

After taking up the chair I arranged a lecture tour (April 1951) around some 12 major US centres of aerodynamic research, all active in areas in which I was interested. For the areas of 'boundary layers and upstream-influence' in particular, I came to know personally during that tour the authors of key experimental and theoretical studies pursued at Caltech (Liepmann *et al.* 1949), MIT (Barry *et al.* 1950), Princeton (Lees 1949; Bogdonoff & Solarski 1951; Lees & Crocco 1952) and Cornell (Kuo 1951) just as I had long known the experimenters at Manchester (Bardsley & Mair 1951; Mair 1952; Johannesen 1952) and NPL (Holder & North 1950; Gadd & Holder 1952) and the Bristol theoreticians (Howarth 1948; Stewartson 1951). During this golden age (1949–1952) of upstream-influence research, a close-knit Anglo-American community—influenced also by papers from Rome (Ferri 1939), Göttingen (Oswatitsch & Wieghardt 1941) and Zürich (Ackeret *et al.* 1946)—felt committed to making sense of all the experimental evidence that disturbances to a supersonic flow can have a substantial upstream influence, through the agency of a boundary layer, even though in inviscid theory a disturbance that leaves the flow supersonic can have no such influence.

An early advance by Liepmann *et al.* (1949) emphasized distinctions between how a laminar and a turbulent layer respond to an incident shock wave, and illustrated them by sketches (figure 1) of 'typical' reflection patterns, with separation absent for the turbulent pattern yet dominating the laminar pattern. However, an admirably systematic study (Barry *et al.* 1950) went beyond the concept of 'typical' patterns and displayed the interaction with boundary layers on a flat surface, at more than one

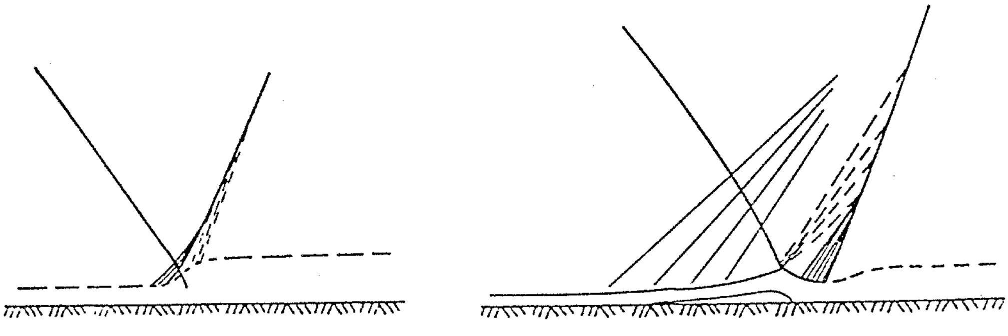

Figure 1. Reflection patterns.

Reynolds number, of a sequence of different shock waves generated by sharp-edged plates at a range of angles (1°, 2°, 3°, 4°, 5° and 6°) to an oncoming supersonic stream. Upstream influence through a laminar layer was always substantial, yet, for the relatively weaker shocks, did not involve separation. They tellingly concluded: 'further theoretical work is necessary. The results presented here may serve as a guide to such work.' Bardsley & Mair (1951), on the other hand, studied details of separation-free reflection of moderately strong shock waves from turbulent layers, in experiments suggested by, and tending to confirm, my own theoretical predictions (see § 2 *d*); yet they also showed that rather strong shock waves do cause even a turbulent layer to separate. Figure 1 is an oversimplification, then, although a major quantitative distinction remains between the tendency of incident shock waves to cause separation of laminar and turbulent layers.

(*b*) *My 1953 argument about two distinct mechanisms of upstream influence*

My comprehensive papers (Lighthill 1953*a*, *b*) were concerned with boundary layers and upstream influence in general, including the upstream effects of any flow deflection generated by a corner (concave or convex) in a wall as well as the upstream effects of a shock wave incident upon a flat wall. From my analysis of the experimental and theoretical papers mentioned above and from my own further theoretical work I concluded that two separate mechanisms exist for upstream influence via a boundary layer.

I also enquired whether they were special to supersonic flow, but received a different answer in each case. Mechanism (i)—originally outlined by Oswatitsch & Wieghardt (1941) and given more precision in Lighthill (1953*b*)—does arise (see § 2 below) from a special property of supersonic flow, and, moreover, fails in subsonic flow.

Mechanism (ii) depends, however (Liepmann *et al.* 1949; Lees 1949; Gadd & Holder 1952), on upstream spreading of a separated-flow region until it is so slender as to cause no further separation ahead of it (see § 3 below). By comparing comprehensive experiments (Mair 1952) on such spreading in supersonic flow with some analogous subsonic data, I demonstrated very broad similarities along with some interesting minor differences in Lighthill (1953*a*). This comparison is sketched in § 3 *a* below; after which, in § 3 *b*, I offer a fuller account of the remarkable discovery by Mair (1952), during his experiments on upstream spreading of separation carried out in the supersonic tunnel of Manchester's Fluid Motion Laboratory, of a wide range of unexpected 'steady and unsteady separated flows', which are especially appropriate to be described (along with an analysis of sources of unsteadiness) at a discussion meeting with this title.

2. Mechanism (i), exclusive to supersonic flow without separation

We have seen (§ 1 *a*) that when a weak shock is incident upon a laminar layer, or a moderately strong shock is incident on a turbulent layer, separation is avoided. It is also absent where a convex corner deflects a supersonic flow, so as to produce an expansion wave outside the boundary layer. The interaction of weak compressive or expansive disturbances with a laminar or turbulent layer, in such cases without separation, is discussed in § 2, with special (although not exclusive) emphasis on how upstream influence may arise.

(*a*) *The Oswatitsch & Wieghardt (1941) mechanism (i) and its relation to the Howarth (1948) approach*

Mechanism (i) for upstream influence (Oswatitsch & Wieghardt 1941) depends, as mentioned in § 1 *b* above, on a special characteristic of flow at Mach number $M_1 > 1$. This is that its deflection by a small angle η yields a pressure change in direct proportion to η.

Specifically, the non-dimensional pressure excess,

$$P = \frac{p - p_1}{\gamma p_1} \tag{2.1}$$

(where p_1 is undisturbed pressure and γ the adiabatic index), becomes

$$P = A_2 \eta, \quad \text{where } A_2 = \frac{M_1^2}{(M_1^2 - 1)^{1/2}}. \tag{2.2}$$

It follows that wall curvature $\mathrm{d}\eta/\mathrm{d}x$ (taken positive for curvature concave to the stream) generates a pressure gradient $\mathrm{d}P/\mathrm{d}x = A_2 \mathrm{d}\eta/\mathrm{d}x$.

Similarly, on a flat wall, any curvature $\mathrm{d}^2\delta_1/\mathrm{d}x^2$ of a boundary layer's displacement thickness contour must give a pressure gradient

$$\frac{\mathrm{d}P}{\mathrm{d}x} = A_2 \frac{\mathrm{d}^2\delta_1}{\mathrm{d}x^2}. \tag{2.3}$$

Yet this gradient, in turn, may be expected to thicken the layer, at a spatial rate

$$\frac{\mathrm{d}\delta_1}{\mathrm{d}x} = A_1 \left(A_2 \frac{\mathrm{d}^2\delta_1}{\mathrm{d}x^2} \right), \tag{2.4}$$

where the coefficient A_1 (rate of thickening per unit gradient of non-dimensional pressure excess), though far from precisely known, is substantially greater for a laminar than for a turbulent layer. Evidently, equation (2.4), with its solutions proportional to $\exp(x/A_1A_2)$, suggests for upstream influence an e-folding distance

$$A_1 A_2. \tag{2.5}$$

An alternative proposal for interpreting upstream influence had been put forward by Howarth (1948). It was based on the idea that upstream influence, prohibited in supersonic flow, may become possible via a boundary layer's subsonic portion. The idea of 'propagation up the subsonic layer' suggested a theoretical approach in which viscosity would be regarded as affecting only the undisturbed distribution of velocity in the boundary layer (including its subsonic portion) but not the disturbances to it. Yet detailed attempts to pursue this approach had run into very serious difficulties in the immediate neighbourhood of the wall. Moreover, after these difficulties were finally resolved, in Lighthill (1953*b*), the Howarth mechanism was found (see below) to become identical with the Oswatitsch & Wieghardt (1941) mechanism (i), although with A_1 now relatively precisely determined.

(*b*) *Resolving difficulties in the Howarth (1948) approach*

The essential hint on how to resolve those difficulties emerged from boundary-layer stability theory, already well established by Tollmien (1929) and Schlichting (1933). According to that theory, small disturbances,

$$[u(y), v(y), 0]\mathrm{e}^{\mathrm{i}k(x-ct)}, \tag{2.6}$$

to a parallel flow $[U(y), 0, 0]$ satisfy, at low Mach number, the well-known Orr–Sommerfeld equation:

$$[U(y) - c](v'' - k^2 v) - U''(y)v = (\nu/\mathrm{i}k)(v'''' - 2k^2 v'' + k^4 v), \tag{2.7}$$

where the right-hand side, proportional to kinematic viscosity ν, is known to be significant only

(a) in a wall layer around where $U(y) = 0$, and

(b) in a critical layer around where $U(y) = c$.

But, if disturbances independent of t are to be represented by equation (2.6), c is necessarily zero, in which case layers (a) and (b) merge into a single inner viscous layer. Also, if the undisturbed flow has zero pressure gradient, $U''(0) = 0$; accordingly, in that thin inner viscous layer, $U(y)$ can be approximated by $U'(0)y$ and ν by its wall value ν_{w}, and equation (2.7) can be written

$$y(v'' - k^2 v) = L^3(v'''' - 2k^2 v'' + k^4 v), \tag{2.8}$$

with

$$L = \left(\frac{\nu_{\mathrm{w}}}{\mathrm{i}kU'(0)}\right)^{1/3}. \tag{2.9}$$

Equation (2.8) possesses a very simple solution, as follows, for small values of k (and we shall see that this solution has good accuracy when $|kL| < 1$, a condition on wavenumber which excludes only features on the very finest scale). Then

$$yv'' = L^3 v'''', \quad \text{with its solution } v'' = A\,\mathrm{Ai}\left(\frac{y}{L}\right) \tag{2.10}$$

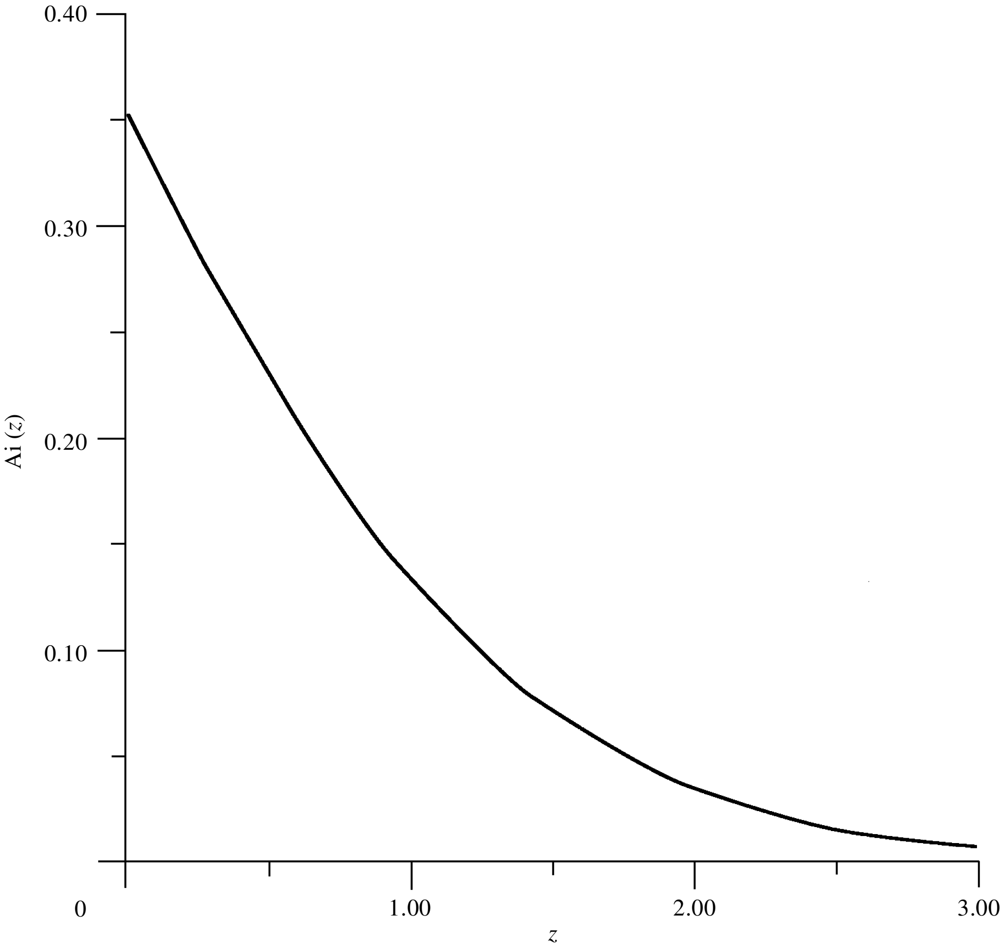

Figure 2. The Airy function.

in terms of the Airy function $\mathrm{Ai}(z)$ plotted in figure 2. Thus, the disturbed viscous stress, proportional to v'', decays exponentially with distance from the wall, and is essentially negligible for $y > 2L$. Integrating twice, we obtain

$$v = A \int_0^y (y - y_1)\, \mathrm{Ai}\left(\frac{y_1}{L}\right) \mathrm{d}y_1, \tag{2.11}$$

where the upper limit of integration can be replaced by ∞ in the essentially inviscid region where v'' has become negligible, giving

$$v = A\left[yL \int_0^\infty \mathrm{Ai}(z)\, \mathrm{d}z - L^2 \int_0^\infty z\, \mathrm{Ai}(z)\, \mathrm{d}z\right]. \tag{2.12}$$

Thus, the disturbance velocity v, in the region where it is governed by inviscid equations, behaves as if the location where it has to vanish is not the wall, $y = 0$,

Table 1. *Location of zero v*

kL	0	± 0.5	± 1.0	± 1.5	± 2.0
$(y/L)_{v=0}$	0.78	0.75	0.70	0.64	0.58

but is given by

$$\frac{y}{L} = \frac{\displaystyle\int_0^\infty z\,\mathrm{Ai}(z)\,\mathrm{d}z}{\displaystyle\int_0^\infty \mathrm{Ai}(z)\,\mathrm{d}z} = 0.78, \tag{2.13}$$

namely the centroid of the area depicted in figure 2. This modified boundary condition on solutions to the inviscid equations is found (see below) to be of the greatest importance.

Lighthill (1953*a*, *b*) also computed exact solutions of the full equation (2.8), determined their exact behaviour for large y/L, and extrapolated that behaviour back to where v vanishes, giving the results in table 1. As indicated earlier, these results suggest that condition (2.13) may be used with confidence whenever $|kL| < 1$.

(*c*) *Perturbations by incident wave and/or wall deflection*

This conclusion, that equation (2.13) specifies the point where an ordinary inviscid boundary condition has to be satisfied by any solution of equations for inviscid disturbances to a boundary layer, is precisely what is required to remove singularities from those equations. Actually, steady inviscid disturbances to a Mach number distribution $[M(y), 0, 0]$ satisfy in terms of η, the flow deflection, and P, the non-dimensional pressure excess (2.1), a pair of equations

$$\frac{\partial \eta}{\partial x} = -M^{-2}(y)\frac{\partial P}{\partial y}, \qquad \frac{\partial \eta}{\partial y} = [M^{-2}(y) - 1]\frac{\partial P}{\partial x}, \tag{2.14}$$

whose singularity in any interval in which $M(y)$ becomes zero is removed if the boundary condition at the wall is replaced by one at the location (2.13) where $M(y)$ takes the value

$$M_2 = 0.78LM'(0). \tag{2.15}$$

Equations (2.14) then simply have to be satisfied in a layer in which $M(y)$ varies between M_2 and the freestream value M_1.

A physical interpretation of equations (2.14) is that the first relates streamline curvature to the centrifugal action of cross-stream pressure gradient, while the second specifies how streamtube-area expansion responds to streamwise pressure gradient (in a manner which changes sign where $M(y) = 1$, as is familiar from the theory of the convergent–divergent nozzle). It is of course precisely this thickening in response to a gradient $\partial P/\partial x$ that the coefficient A_1 of § 2 *a* is supposed to represent, and the coefficient in square brackets integrated across the boundary layer (that is, from $M(y) = M_2$ to $M(y) = M_1$) will emerge (see below) as a first approximation to A_1.

After η is eliminated from equations (2.14), while P is expressed as an integral

$$P = \int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}kx}\Pi(k, y)\,\mathrm{d}k, \tag{2.16}$$

the Fourier transform Π is found to satisfy the ordinary differential equation

$$\frac{\mathrm{d}}{\mathrm{d}y}\left[M^{-2}(y)\frac{\mathrm{d}\Pi}{\mathrm{d}y}\right] = k^2[M^{-2}(y)-1]\Pi, \tag{2.17}$$

together with boundary conditions as follows. Whenever wall deflection, with a distribution

$$\eta = \int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}kx}H(k)\,\mathrm{d}k, \tag{2.18}$$

is responsible for generating some or all of the disturbance, then the first of equations (2.14) gives the wall boundary condition as

$$\Pi_y(k,0) = -M_2^2\mathrm{i}kH(k), \tag{2.19}$$

because $M(y)$ takes the value M_2, given by equation (2.15), at the location (here redefined as $y=0$) where the inviscid boundary condition has to be satisfied. Also, at the edge $y=\delta$ of the boundary layer, disturbances take the simple form of small perturbations to a uniform stream of Mach number $M_1 > 1$; thus,

$$P = f(x+\beta y) + g(x-\beta y), \quad \text{where } \beta = (M_1^2-1)^{1/2}. \tag{2.20}$$

If, here, the functions f (incident wave) and g (emitted wave) have Fourier transforms $F(k)$ and $G(k)$, then

$$P = \int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}kx}[F(k)\mathrm{e}^{\mathrm{i}k\beta y} + G(k)\mathrm{e}^{-\mathrm{i}k\beta y}]\,\mathrm{d}k, \tag{2.21}$$

so that its Fourier transform $\Pi(k,y)$ takes, where $y=\delta$, the square-bracketed form given here in terms of a known incident wave and the unknown emitted wave. Elimination of the latter then gives a boundary condition

$$\Pi_y(k,\delta) + \mathrm{i}k\beta\Pi(k,\delta) = 2\mathrm{i}k\beta\mathrm{e}^{\mathrm{i}k\beta\delta}F(k). \tag{2.22}$$

Now, any solution of the second-order equation (2.17) can be expressed as a linear combination of two independent solutions, $Q(k,y)$ and $T(k,y)$, satisfying, at $y=0$, the conditions

$$Q(k,0)=1 \quad \text{and} \quad Q_y(k,0)=0; \qquad T(k,0)=0 \quad \text{and} \quad T_y(k,0)=1. \tag{2.23}$$

In particular, the solution which satisfies the boundary conditions (2.19) and (2.22) can be written as

$$\begin{aligned}\Pi = {} & 2\mathrm{i}k\beta\mathrm{e}^{\mathrm{i}k\beta\delta}F(k)\frac{Q(k,y)}{Q_y(k,\delta)+\mathrm{i}k\beta Q(k,\delta)} \\ & - M_2^2\mathrm{i}kH(k)\left\{T(k,y) - \frac{T_y(k,\delta)+\mathrm{i}k\beta T(k,\delta)}{Q_y(k,\delta)+\mathrm{i}k\beta Q(k,\delta)}Q(k,y)\right\},\end{aligned} \tag{2.24}$$

where the first line represents the effect of an incident wave on a flat wall (it satisfies (2.19) with $H(k)=0$), and the second line represents the effect of wall deflection without any incident wave (it satisfies (2.22) with $F(k)=0$); indeed, only in the simultaneous presence of an incident wave and wall deflection would $\Pi(k,y)$ take the two-line form (2.24).

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For completeness we may also note an expression

$$G(k) = \frac{-\mathrm{e}^{2\mathrm{i}k\beta\delta}[Q_y(k,\delta) - \mathrm{i}k\beta Q(k,\delta)]F(k) + M_1^2 \mathrm{i}k\mathrm{e}^{\mathrm{i}k\beta\delta}H(k)}{Q_y(k,\delta) + \mathrm{i}k\beta Q(k,\delta)} \tag{2.25}$$

for the Fourier transform of the emitted wave; which, of course, can be viewed as a reflection of any incident wave and/or as an emission produced by any wall deflection. Results (2.24) and (2.25) are used in § 2 *d* to make the e-folding distance of upstream influence via a boundary layer relatively precise, and (more briefly) to discuss some fine-scale features of the emitted wave and of the wall pressure distribution.

(*d*) *A relatively precise form of mechanism (i) for upstream influence*

A key to upstream influence is the presence of a common denominator throughout expressions (2.24) and (2.25), which have singularities of 'simple-pole' type wherever that denominator vanishes. As $x \to -\infty$, the asymptotic behaviour of the Fourier integral (2.16) for P is found by moving the path of integration downwards in the complex k-plane until it crosses the first such pole. If then κ_1 is the least positive number such that the denominator vanishes for $k = -\mathrm{i}\kappa_1$, giving

$$Q_y(-\mathrm{i}\kappa_1, \delta) + \kappa_1 \beta Q(-\mathrm{i}\kappa_1, \delta) = 0, \tag{2.26}$$

all disturbances vary asymptotically

$$\text{like } \mathrm{e}^{\kappa_1 x}, \quad \text{as } x \to -\infty. \tag{2.27}$$

In many problems of mechanics of thin plates and thin layers, a similar asymptotic behaviour (2.27) is found, with κ_1 as the least eigenvalue satisfying a condition analogous to (2.26). St Venant's Principle (see § 1 above) states that, 'in general', the e-folding distance, κ_1^{-1}, for decay of disturbances will hardly exceed the layer thickness; thus, the 'intriguing departure' from it to which I referred in my chair interview lies in an unusual smallness of the eigenvalue κ_1, which in turn arises because the singularity of equations (2.14) is approached closely at $M(y) = M_2$, the effective wall Mach number (2.15), where for $k = -\mathrm{i}\kappa_1$, the expression (2.9) for L becomes

$$L = \left(\frac{\nu_{\mathrm{w}}}{\kappa_1 U'(0)}\right)^{1/3}. \tag{2.28}$$

Accordingly, although large-k solutions are used later to discuss fine-scale features, a relatively precise determination of upstream influence comes from small-k solutions. For $k = 0$, equation (2.17) under boundary condition (2.23) has the solution $Q = 1$, whose substitution on the right-hand side of (2.17) gives, for small k, the approximate solutions

$$Q_y = k^2 M^2(y) \int_0^y [M^{-2}(z) - 1]\,\mathrm{d}z, \quad Q = 1 + k^2 \int_0^y M^2(y)\,\mathrm{d}y \int_0^y [M^{-2}(z) - 1]\,\mathrm{d}z. \tag{2.29}$$

To this approximation, equation (2.26) for κ_1 becomes

$$-\kappa_1^2 M_1^2 \int_0^\delta [M^{-2}(z) - 1]\,\mathrm{d}z + \kappa_1 \beta \left\{1 - \kappa_1^2 \int_0^\delta M^2(y)\,\mathrm{d}y \int_0^y [M^{-2}(z) - 1]\,\mathrm{d}z\right\} = 0, \tag{2.30}$$

where the smallness of M_2, the value of $M(z)$ at the lower limit of the first integral, leads directly to the smallness of κ_1.

Actually, equation (2.30) includes terms in κ_1, κ_1^2 and κ_1^3, and omission of the last of these gives, to first order,

$$\int_0^\delta [M^{-2}(z) - 1]\,\mathrm{d}z = \frac{\beta}{\kappa_1 M_1^2}. \tag{2.31}$$

It may, however, be worthwhile to also study the last term (in κ_1^3), because its inner integral,

$$\int_0^y [M^{-2}(z) - 1]\,\mathrm{d}z, \tag{2.32}$$

has the same lower limit (where $M(0)$ takes the small value M_2) as integral (2.31); so that the two integrals may be approximately equal except where y is small compared with δ, when, on the other hand, the value of (2.32) is unimportant, because, in equation (2.30), it is multiplied by the small factor $M^2(y)$. Equating the integrals (2.31) and (2.32) in condition (2.30) turns it into a useful second-order expression for the e-folding distance

$$\kappa_1^{-1} = \frac{M_1^2}{\beta}\int_0^\delta [M^{-2}(y) - 1]\,\mathrm{d}y + \frac{\beta}{M_1^2}\int_0^\delta M^2(y)\,\mathrm{d}y, \tag{2.33}$$

which is 'robust' in its indifference to how the edge, $y = \delta$, of the boundary layer is defined (a change to $\delta + \varepsilon$ alters the second term by $+\beta\varepsilon$ and the first by $-\beta\varepsilon$).

The first term on the right-hand side of (2.33) takes the Oswatitsch & Wieghardt (1941) form $A_1 A_2$ for the e-folding factor, with A_2 specified by equation (2.2) and A_1 as an integral of the square-bracketed coefficient that, in (2.14), relates streamtube-area expansion to the gradient $\partial P/\partial x$. That first term is, however, given greater precision by addition of the (smaller) second term associated with centrifugal-force effects.

Equation (2.33) has to be solved for κ_1, taking into account the dependence on κ_1 of expressions (2.28) for L and (2.15) for M_2. Here, I show this only for one case treated by Lighthill (1953*a*, *b*), using data for a zero-pressure-gradient laminar layer with its 'thickness' δ defined as distance from the wall, where the supersonic mainstream velocity is attained to within 5%. Then $(\kappa_1\delta)^{-1}$, the ratio of e-folding distance to layer thickness, satisfies, to second order, the condition

$$(\kappa_1\delta)^{-1} = (\kappa_1\delta)^{1/3}\frac{1.3}{\beta}\left(\frac{T_\mathrm{w}}{T_1}\right) Re^{1/6} - \frac{M_1^2 + 2}{2\beta}\left(\frac{T_\mathrm{w}}{T_1}\right)^{0.7}, \tag{2.34}$$

where Re is Reynolds number based on distance from the leading edge, and an appearance of the ratio T_w/T_1 of wall temperature to freestream temperature can be interpreted as due to any effect of pressure gradient on layer thickening being enhanced near a hot wall. For air in the zero-heat-transfer case, T_w/T_1 is about $1+0.17M^2$, when solutions of the algebraic equation (2.34) for $(\kappa_1\delta)^{-1}$ are as plotted in figure 3 for values 10^5 and 10^6 of Re. At least when $1 < M < 2$, the e-folding distance for upstream influence is seen to be of the order of five layer thicknesses.

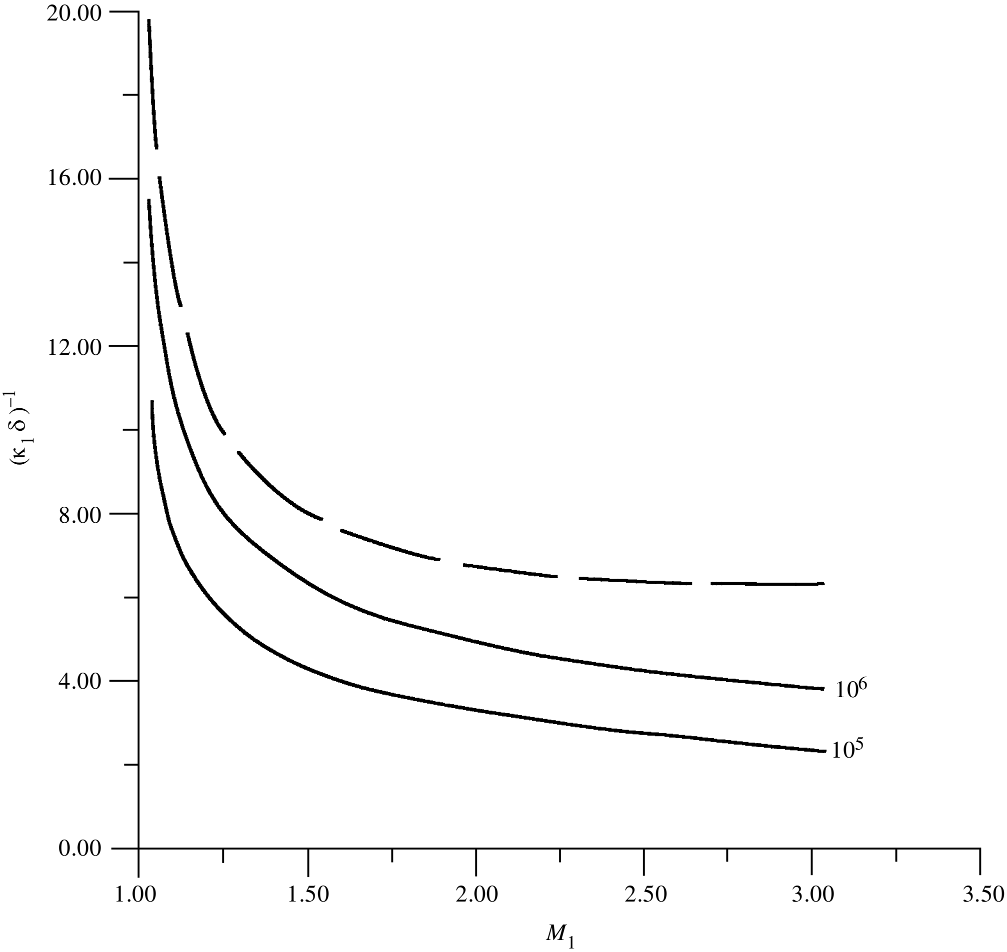

Figure 3. Solutions of (2.34) (solid curves), for $Re = 10^5$, 10^6, and of (2.35) (dashed curve).

On the other hand, the simple first-order solution

$$(\kappa_1 \delta)^{-1} = \left[\frac{1.3}{\beta}\left(\frac{T_{\mathrm{w}}}{T_1}\right)\right]^{3/4} Re^{1/8}, \tag{2.35}$$

obtained from (2.34) by neglecting its last term, is seen in figure 3 to give rather poor accuracy, even at the highest value of *Re* (around 10^6) for which the boundary layer is laminar. This casts some doubt on the practical wisdom of later approaches based on treating $Re^{1/8}$ as a large number!

Instead of using such an approach, Lighthill (1953*a*, *b*) was content with an *a posteriori* check on the assumptions he had made. When the calculation had been completed, the value obtained for κ_1 was used in equations (2.28) and (2.15) to compute M_2, whose continued smallness (less than or equal to 0.3) confirmed the correctness of treating the inner viscous layer as thin and of using a low-Mach-number equation (2.7) within it.

The small-k solutions of (2.17) have dominated this discussion on upstream influence, which, without separation, is seen to be absent for turbulent layers yet substantial for laminar layers, especially at moderate $M_1 > 1$. Very briefly, however, the large-k (WKB–Langer) solutions show that

(a) an incident wave

$$f(x + \beta y) = \int_{-\infty}^{\infty} e^{ikx} F(k) e^{ik\beta y}\, dk$$

yields a reflection,

$$g(x - \beta y) = \int_{-\infty}^{\infty} e^{ikx} G(k) e^{-ik\beta y}\, dk,$$

with large-k behaviour $G(k) \sim e^{2ik\theta}(i\,\mathrm{sgn}\,k)F(k)$, where

$$\theta = \beta\delta - \int_{y_1}^{\delta} [M^2(y) - 1]^{1/2}\, dy$$

is a phase shift due to cusped reflection at the sonic line $y = y_1$, and where, in addition, the $(i\,\mathrm{sgn}\,k)$ factor causes a pressure 'step' in the incident wave to be reflected locally as a 'ridge' (for a laminar or turbulent layer);

(b) nonetheless, the wall pressure rises monotonically; while

(c) at a convex corner (with no incident wave) an expansion wave with monotonic pressure drop is emitted, and, also, the wall pressure falls monotonically;

all of this being smoothed over a distance

$$\sigma = \int_0^{y_1} [1 - M^2(y)]^{1/2}\, dy.$$

3. Mechanism (ii): on upstream spreading of separation

(a) *Comparison*

Now I move forward to mechanism (ii): upstream spreading of a separated-flow region. Unlike mechanism (i), this can work even in subsonic flow, where, admittedly, discontinuous incident waves are impossible, and yet wall deflection can be discontinuous (and have an upstream influence). I discovered, in plate 19(b) of Goldstein (1938), a photograph (unpublished elsewhere) by W. S. Farren of low-speed flow up a step, in which a separated-flow region has spread upstream to become so slender that boundary-layer separation is just initiated at its leading cusp. Here, the external flow can be determined as a simple free streamline problem.

The upper half of this symmetrical flow up a step is derived using the complex coordinate $z = x + iy$ and potential $w = \phi + i\psi$, where

$$\Omega = \ln \frac{dw}{dz} = \ln q - i\eta,$$

with flow speed q and direction η, fills this domain. With

$$k = \frac{\pi/2}{\ln(u/v)},$$

and S as the length of the streamline BC, the conformal mapping is

$$w = VS \tanh^2\left[k\left(\eta + \mathrm{i}\ln\frac{q}{v}\right)\right].$$

So the streamline BC has the 'intrinsic' equation

$$s = S\tanh^2(k\eta),$$

and the centreline velocity q satisfies

$$x = -S\int_0^{\ln(q/v)} \mathrm{e}^{-t}\,\mathrm{d}\{\tan^2(kt)\}.$$

The experiment corresponds to $k = 4.6$, with boundary-layer separation after a mainstream-velocity fall by 9% (cf. 12% (Howarth 1938) for linear retardation), giving $b/h = 4.6$, $\ell/h = 5.7$ and laminar separation at

$$C_p = \frac{p - p_1}{\frac{1}{2}\rho U^2} = 0.17.$$

(The main differences in supersonic flow are that shock-initiated separation gives a straight free streamline, and it occurs at a lower C_p (≈ 0.08 at $Re = 10^5$, and less at higher Re).)

(*b*) *On Mair's experiments*

Here I leave my own old paper and devote the rest of my current paper to W. A. Mair's marvellously complex 1952 experiments involving both steady and unsteady separated flows. Mair noted the following.

> If a blunt body of revolution is fitted with a slender probe at the nose and placed in a supersonic airstream, there is an interaction between the shock wave in front of the body and the boundary layer on the probe.

Further,

> Schlieren photographs were also taken of the corresponding two-dimensional flow, using a thin plate clamped between two thicker flat-nosed plates.

Mair's (1952) principal work was on axisymmetric blunt-nosed bodies in supersonic flow ($M = 1.96$), where a thin probe converts a high-drag detached-shock regime (his fig. 4) into a regime (his fig. 7) with conical shock and much lower drag (an ultra-lightweight 'fairing'). The steady conical-shock regime was achieved for probe lengths $1.3d$ to $2.1d$ (flat nose), and $0.3d$ to $1.65d$ (hemispherical), where d is the body diameter. In addition, two types of unsteady separated flow were observed:

(I) *irregular fluctuation*; whenever the probe length exceeded $2.1d$ (flat nose) or $1.65d$ (hemispherical), the separation point was downstream of the probe shoulder (see below) and varied in a highly intermittent fashion; and

(II) *a regular oscillation* (at *ca.* 6 kHz) was observed (for the flat nose only) with probe lengths from $0.7d$ to $1.3d$.

Concerning (I), the irregular fluctuation with long probes, the intermittent variation of the separation point extends over a wide spectrum. Thus, fluctuation components of *ca.* 10 Hz were directly visible (on a screen), yet other components were in kHz.

Mair's (1952) fig. 5 shows flash photographs (instantaneous patterns) with separation 'presumably' laminar in (a) and (c), turbulent in (b) and (d), due to a variably disturbed mainstream; a much lower pressure jump being needed to separate a laminar layer. As confirmation, flows (e) and (f), with transition fixed (by a thin wire ring), are steady.

Rapid upstream movement of the separation point (at approximately 0.4 times the speed of sound) was shown in (a) and (c) by the angle to the axis of the conical shock wave (*ca.* 25° as against 30.7° Mach angle). Similarly, in Mair's fig. 6, for probes not quite so long, (a), (b) and (c) show upstream movement, but (d) downstream! (The shock angle exceeds the value for steady conical flow.)

Mair also found (II) regular oscillation at *ca.* 6 kHz on a flat-nosed body with probe lengths from $0.7d$ to $1.3d$; and identified its nature by first taking numerous flash photographs and then sequencing a selection: his fig. 12(a) with two bow waves (upstream wave moving aft, to yield in (b) a single 'split' bow wave), while separation moves upstream; (b) 50 μs later, with separation at the probe nose giving a large dead-air region and incipient strong shock; (c) 30 μs later, with larger dead-air region, with probe bow shock extended and with annular body bow wave; (d) 20 μs later, with annular wave moving aft (the origin of the emission of weak downstream-moving shocks); (e) 40 μs later, and (f) 20 μs later still, both with the dead-air region contracting, and its shock moving aft, while a new body bow wave appears, to give (a) soon after! Finally, some corresponding two-dimensional flows: his fig. 15(a) thick plate, no thin plate; (b), (c) thin plates of length $0.55d$ (same bow wave) and $2.5d$ (steady wedge-shaped dead-air region); (d), (e), (f) with intermediate lengths $1.5d$ and $2.0d$ showing unsteady flows, which need not be symmetrical!

4. Conclusion

Forty-five years ago, boundary layers and upstream influence were alive and well and living in Manchester (*inter alia*)!

5. Comment

Sir James completed, by hand, the abstract, § 1 and most of § 2 before his death. The remainder of the paper was taken (by Beryl Lankester and Frank Smith) from the handwritten transparencies of Sir James's talk (see Preface, this issue), as faithfully as possible, consistent with clarity.

References

Ackeret, J., Feldmann, F. & Rott, N. 1946 Mitt. Inst. Aerodyn. Zurich, no.10. Or Tech. Memo. Nat. Adv. Comm. Aero., Wash., no.1113 (1947).

Bardsley, O. & Mair, W. A. 1951 The interaction between an oblique shock-wave and a turbulent boundary layer. *Phil. Mag.* **42**, 29.

Barry, F. W., Shapiro, A. H. & Neumann, E. P. 1950 The interaction of shock waves with boundary layers on a flat surface. *J. Aero. Sci.* **18**, 229.

Bogdonoff, S. M. & Solarski, A. H. 1951 Aero. Engng Lab. Report no. 184 (Princeton, NJ).

Ferri, A. 1939 Atti Guidonia, no. 17 (Rome).

Gadd, G. E. & Holder, D. W. 1952 The interaction of an oblique shock wave with the boundary layer on a flat plate. Part I. Results for $M = 2$. Aero. Res. Council Report no. 14848.

Goldstein, S. 1938 *Modern developments in fluid dynamics*. Oxford University Press.

Holder, D. W. & North, R. J. 1950 Observations of the interaction between the shock waves and boundary layers at the trailing edges of airfoils in supersonic flow. Aero. Res. Council Report no. 13621.

Howarth, L. 1938 On the solution of the laminar boundary layer equations. *Proc. R. Soc. Lond.* A **164**, 547–579.

Howarth, L. 1948 *Proc. Camb. Phil. Soc.* **44**, 380.

Johannesen, N. 1952 *Phil. Mag.* **43**, 568.

Kuo, Y. H. 1951 *On the reflection of a weak shock wave from a boundary layer along a flat plate.* Ithaca, NY: Cornell University.

Lees, L. 1949 Aero. Engng Lab. Report no. 143 (Princeton, NJ).

Lees, L. & Crocco, L. 1952 Aero. Engng Lab. Report no. 195 (Princeton, NJ).

Liepmann, H. W., Roshko, A. & Dhawan, S. 1949 On the reflection of shock waves from boundary layers. Galcit Report (Pasadena, CA). Or Tech. Memo. Nat. Adv. Comm. Aero., Wash., no. 2334 (1951).

Lighthill, M. J. 1953*a* On boundary layers and upstream influence. I. A comparison between subsonic and supersonic flows. *Proc. R. Soc. Lond.* A **217**, 344–357.

Lighthill, M. J. 1953*b* On boundary layers and upstream influence. II. Supersonic flows without separation. *Proc. R. Soc. Lond.* A **217**, 478–507.

Mair, W. A. 1952 *Phil. Mag.* **43**, 695.

Oswatitsch, K & Wieghardt, K. 1941 German wartime report. Reprinted as Tech. Memo. Nat. Adv. Comm. Aero., Wash., no. 1189 (1948).

Schlichting, H. 1933 Zur Entstehung der Turbulenz bei der Plattenströmung. *Nachr. Ges. Wiss. Göttingen, Math.-Phys Kl.* 181–208.

Stewartson, K. 1951 *Proc. Camb. Phil. Soc.* **47**, 545.

Tollmien, W. 1929 Über die Entstehung der Turbulenz. Nachr. Ges. Wiss. Göttingen 21–44. Translated as 'The production of turbulence', Tech. Memo. Nat. adv. Comm. Aero., Wash., no. 609.